

THE INTEGRAL (ORBIFOLD) CHOW RING OF TORIC DELIGNE-MUMFORD STACKS

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ABSTRACT. In this paper we study the integral Chow ring of toric Deligne-Mumford stacks. We prove that the integral Chow ring of a semi-projective toric Deligne-Mumford stack is isomorphic to the Stanley-Reisner ring of the associated stacky fan. The integral orbifold Chow ring is also computed. Our results are illustrated with several examples.

1. INTRODUCTION

Chow groups with integer coefficients of algebraic stacks were defined by Edidin and Graham [EG] and Kresch [Kr], using Totaro's idea [To] of integral Chow ring of classifying spaces. In [EG], the authors constructed an intersection theory of stack quotient $[X/G]$ of a quasi-projective variety X by an algebraic group G . In the case of Deligne-Mumford stacks, the authors proved that the equivariant Chow ring $A_G^*(X)$ with integer coefficients is isomorphic to the integral Chow ring of the quotient stack $[X/G]$.

Toric Deligne-Mumford stacks were introduced by Borisov, Chen and Smith [BCS] via a generalization of the quotient construction [Cox] of simplicial toric varieties. A construction of toric stacks using logarithmic geometry can be found in [Iwa1]. The purpose of this paper is to compute the (orbifold) Chow ring with integer coefficients of a toric Deligne-Mumford stack. A toric Deligne-Mumford stack is defined in terms of a stacky fan $\Sigma = (N, \Sigma, \beta)$, where N is a finitely generated abelian group, $\Sigma \subset N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ is a simplicial fan and $\beta : \mathbb{Z}^n \rightarrow N$ is a map determined by the elements $\{b_1, \dots, b_n\}$ in N . By assumption, β has finite cokernel and $\{\bar{b}_1, \dots, \bar{b}_n\}$ generate the simplicial fan Σ , where \bar{b}_i is the image of b_i under the natural map $N \rightarrow N_{\mathbb{Q}}$. The toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ associated to Σ is defined to be the quotient stack $[Z/G]$, where Z is the open subvariety $\mathbb{C}^n \setminus \mathbb{V}(J_{\Sigma})$, J_{Σ} is the irrelevant ideal of the fan, G is the product of an algebraic torus and a finite abelian group. The G -action on Z is given via a group homomorphism $\alpha : G \rightarrow (\mathbb{C}^*)^n$, where α is obtained by taking $\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ functor to the Gale dual $\beta^{\vee} : \mathbb{Z}^n \rightarrow N^{\vee}$ of β and $G = \text{Hom}_{\mathbb{Z}}(N^{\vee}, \mathbb{C}^*)$.

Each ray ρ_i in the fan Σ gives a line bundle \mathcal{L}_i over $\mathcal{X}(\Sigma)$ which is defined by the quotient $Z \times \mathbb{C}/G$ and the action of G on \mathbb{C} is through the i -th component of the map α . The Picard group of $\mathcal{X}(\Sigma)$ is seen to be isomorphic to N^{\vee} .

Every stacky fan Σ has an underlying *reduced* stacky fan $\Sigma_{\text{red}} = (\overline{N}, \Sigma, \overline{\beta})$, where $\overline{N} = N/\text{torsion}$, $\overline{\beta} : \mathbb{Z}^n \rightarrow \overline{N}$ is the natural projection given by the vectors $\{\overline{b}_1, \dots, \overline{b}_n\} \subseteq \overline{N}$. The toric Deligne-Mumford stack $\mathcal{X}(\Sigma_{\text{red}})$ is a toric orbifold. By construction $\mathcal{X}(\Sigma_{\text{red}}) = [Z/\overline{G}]$, where $\overline{G} = \text{Hom}_{\mathbb{Z}}(\overline{N}^{\vee}, \mathbb{C}^*)$ and \overline{N}^{\vee} is the Gale dual $\overline{\beta}^{\vee} : \mathbb{Z}^n \rightarrow \overline{N}^{\vee}$ of the map $\overline{\beta}$. The stack $\mathcal{X}(\Sigma_{\text{red}})$ can be obtained by the rigidification construction (see e.g. [ACV]). Each ray ρ_i in the fan Σ also gives a line bundle L_i over the toric orbifold which corresponds to the divisor D_i . This line bundle L_i also has a quotient construction $Z \times \mathbb{C}/\overline{G}$ where \overline{G} acts on \mathbb{C} via the i -th component of the map $\overline{\alpha} : \overline{G} \rightarrow (\mathbb{C}^*)^n$, obtained by taking $\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to the map $\overline{\beta}^{\vee}$.

Since N is a finitely generated abelian group, we write it as the invariant factor form

$$N = \mathbb{Z}^d \oplus \mathbb{Z}_{m_1} \oplus \dots \oplus \mathbb{Z}_{m_r},$$

where $m_1|m_2|\dots|m_r$. Let Σ be a stacky fan. Suppose that the map β generates the torsion part of N , then $r \leq n - d$. We prove that the toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ is a nontrivial $\mu = \mu_{m_1} \times \dots \times \mu_{m_r}$ -gerbe over the toric orbifold $\mathcal{X}(\Sigma_{\text{red}})$ obtained as the stack of roots of line bundles M_i . In the Picard group of the toric orbifold, there exist $n - d$ line bundles M_1, \dots, M_{n-d} such that the $\mathcal{X}(\Sigma)$ can be constructed as μ_{m_i} -gerbes over the line bundle M_i for $i = 1, \dots, n - d$. These $n - d$ line bundles form the canonical generators of the Picard group.

Iwanari [Iwa1], [Iwa2] proved that the integral Chow ring of the toric orbifold $\mathcal{X}(\Sigma_{\text{red}})$ is isomorphic to the Stanley-Reisner ring $SR(\Sigma_{\text{red}})$ of the reduced stacky fan Σ_{red} , where

$$(1.1) \quad SR(\Sigma_{\text{red}}) := \frac{\mathbb{Z}[x_i : \rho_i \in \Sigma(1)]}{(I_{\Sigma} + C(\Sigma_{\text{red}}))},$$

the ideal I_{Σ} is generated by

$$(1.2) \quad \{x_{i_1} \cdots x_{i_k} : \rho_{i_1} + \dots + \rho_{i_k} \notin \Sigma\},$$

and $C(\Sigma_{\text{red}})$ is the ideal generated by linear relations:

$$(1.3) \quad \left(\sum_{i=1}^n \theta(b_i) x_i \right)_{\theta \in N^*}.$$

In the toric orbifold case we have that

$$\frac{\mathbb{Z}[x_i : \rho_i \in \Sigma(1)]}{(C(\Sigma_{\text{red}}))} \cong \overline{N}^{\vee},$$

where \overline{N}^{\vee} is the Picard group of the toric orbifold. Let t_1, \dots, t_{n-d} be the canonical generators of the Picard group of $\mathcal{X}(\Sigma_{\text{red}})$. Then t_i 's generate x_i 's and x_i 's generate t_i 's, let $\mathbf{x} = (x_i)$, $\mathbf{t} = (t_i)$ be column vectors, then there exist integral matrices $A = (a_{i,j})$ and $C = (c_{i,j})$ such that $\mathbf{x} = A\mathbf{t}$, and $\mathbf{t} = C\mathbf{x}$. Let $\tilde{\mathbf{x}} = (\tilde{x}_i)$

be a column vector and

$$M = \begin{bmatrix} m_1 & & \\ & \ddots & \\ & & m_{n-d} \end{bmatrix}$$

a diagonal matrix such that

$$(1.4) \quad \tilde{\mathbf{x}} := AM\mathbf{t} = AMC\mathbf{x} = E\mathbf{x},$$

where $E = AMC$ is a $n \times n$ integral matrix. So every \tilde{x}_i is a integral linear combination of x_i 's. Let \mathcal{I}_Σ be an ideal in $\mathbb{Z}[x_i : \rho_i \in \Sigma(1)]$ generated by

$$(1.5) \quad \{\tilde{x}_{i_1} \cdots \tilde{x}_{i_k} : \rho_{i_1} + \cdots + \rho_{i_k} \notin \Sigma\}.$$

We define the Stanley-Reisner ring $SR(\Sigma)$ of the stacky fan Σ as follows:

$$(1.6) \quad SR(\Sigma) := \frac{\mathbb{Z}[x_i : \rho_i \in \Sigma(1)]}{(\mathcal{I}_\Sigma + C(\Sigma))},$$

where $C(\Sigma)$ is the same as the ideal $C(\Sigma_{\text{red}})$ in (1.3).

Let $A^*(\mathcal{X}(\Sigma), \mathbb{Z})$ be the integral Chow ring of the toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$. Then we have:

Theorem 1.1. *Let $\mathcal{X}(\Sigma)$ be a toric Deligne-Mumford stack with semi-projective coarse moduli space. Suppose that in the map β of the stacky fan Σ , the vectors b_1, \dots, b_n generate the torsion part of N . Then there is an isomorphism of rings:*

$$A^*(\mathcal{X}(\Sigma), \mathbb{Z}) \cong SR(\Sigma).$$

Since the toric Deligne-Mumford stack is a nontrivial μ -gerbe constructed from a sequence of root gerbes of line bundles, Theorem 1.1 is proved by calculating Chow rings of root gerbes in terms of those of the bases and applying the result of Iwanari [Iwa2], see Section 4.

The rational orbifold Chow ring of projective toric Deligne-Mumford stacks are computed in [BCS]. Their result was generalized to semi-projective case in [JT]. The orbifold Chow ring is isomorphic to the deformed ring of the fan. Using Theorem 1.1 we compute the integral orbifold Chow ring for any semi-projective toric Deligne-Mumford stacks. For a stacky fan Σ , let σ be a cone in Σ , define $\text{Box}(\sigma)$ to be all the elements $v \in N$ such that $\bar{v} = \sum_{\rho_j \subset \sigma} \alpha_j \bar{b}_j \in N$ for $0 \leq \alpha_j < 1$. Let $\text{Box}(\Sigma)$ be the disjoint union of all $\text{Box}(\sigma)$ for $\sigma \subset \Sigma$. The set $\text{Box}(\Sigma)$ is finite and may be listed as $\{v_1, \dots, v_k\}$.

Now we consider the integral orbifold Chow ring of the toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$. First we introduce some notations. Let $\tilde{\mathbf{y}} = (\tilde{y}^{b_i})$ and $\mathbf{y} = (y^{b_i})$ for $1 \leq i \leq n$ be column vectors such that

$$\tilde{\mathbf{y}} = E\mathbf{y},$$

where the matrix $E = AMC$ is the same as the matrix in (1.4). We introduce the ring:

$$S_\Sigma := \frac{\mathbb{Z}[y^{b_i} : \rho_i \in \Sigma(1)]}{\mathcal{I}_\Sigma},$$

where y is a formal variable and the ideal \mathcal{I}_Σ is the ideal generated by the elements in (1.5) replacing \tilde{x}_i by \tilde{y}^{b_i} , i.e. by the elements

$$(1.7) \quad \{\tilde{y}^{b_{i_1}} \cdots \tilde{y}^{b_{i_k}} : \rho_{i_1} + \cdots + \rho_{i_k} \notin \Sigma\}.$$

We define a ring

$$(1.8) \quad \mathbb{Z}[\Sigma] = S_\Sigma[y^{v_1}, \dots, y^{v_k}],$$

which is a ring over the Stanley-Reisner ring. The product is defined as follows: For any two $v_1, v_2 \in \text{Box}(\Sigma)$, let $v_1 + v_2 = \sum_{\rho_j \subset \sigma(\bar{v}_1, \bar{v}_2)} a_j b_j$ and let I be the set of rays ρ_i such that $a_i > 1$, J the set of rays ρ_j such that ρ_j belongs to $\sigma(\bar{v}_1), \sigma(\bar{v}_2)$, but not $\sigma(\bar{v}_3)$. Then

$$(1.9) \quad y^{v_1} \cdot y^{v_2} := \begin{cases} y^{\bar{v}_3} \prod_{i \in I} \tilde{y}^{b_i} \cdot \prod_{i \in J} \tilde{y}^{b_i} & \text{if there is a cone } \sigma \in \Sigma \text{ such that } \bar{v}_1 \in \sigma, \bar{v}_2 \in \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\text{Cir}(\Sigma)$ be the ideal generated by the elements in (1.3) replacing x_i by y^{b_i} , i.e. by the linear relations:

$$(1.10) \quad \left(\sum_{i=1}^n \theta(b_i) y^{b_i} \right)_{\theta \in N^*}.$$

Let $A_{orb}^*(\mathcal{X}(\Sigma), \mathbb{Z})$ be the integral orbifold Chow ring of the toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$. Then we have:

Theorem 1.2. *Let $\mathcal{X}(\Sigma)$ be a toric Deligne-Mumford stack associated to the stacky fan Σ such that the coarse moduli space is semi-projective and the map β generates the torsion part of N in the stacky fan Σ . Then we have an isomorphism of graded rings:*

$$A_{orb}^*(\mathcal{X}(\Sigma), \mathbb{Z}) \cong \frac{\mathbb{Z}[\Sigma]}{\text{Cir}(\Sigma)}.$$

This is the first nontrivial examples of the formula for integral orbifold Chow rings. Using Theorem 1.1, the proof of Theorem 1.2 is similar to [BCS], except that we work with integer coefficients.

This paper is organized as follows. In Section 2 we recall the construction of Gale duality for finitely generated abelian groups in [BCS] and use it to study toric Deligne-Mumford stacks. We give a construction of toric Deligne-Mumford stacks in this section. In Section 3 we discuss line bundles over toric Deligne-Mumford stacks and determine the Picard group of toric Deligne-Mumford stacks. In Section 4 we study the integral Chow ring of toric Deligne-Mumford stacks. We prove that the integral Chow ring of a toric Deligne-Mumford stack is isomorphic to the Stanley-Reisner ring of its stacky fan. We study the integral orbifold Chow ring in Section 5, and in Section 6 we compute some examples.

Conventions. In this paper we work entirely algebraically over the field of complex numbers. Chow rings and orbifold Chow rings are taken with integer coefficients. By an orbifold we mean a smooth Deligne-Mumford stack with trivial generic stabilizer.

We use N^* to denote the dual $\text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ and \mathbb{C}^* the multiplication group $\mathbb{C} - \{0\}$. We denote by $N \rightarrow \overline{N}$ the natural map modulo torsion. Since N is a finitely generated abelian group, we write

$$N = \mathbb{Z}^d \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r},$$

where $m_1 | m_2 | \cdots | m_r$. This is called the invariant factor decomposition.

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2. TRIVIAL AND NONTRIVIAL GERBES

In this section we first recall the construction of Gale duality for finitely generated abelian groups. We use the properties of Gale duality to classify toric Deligne-Mumford stacks as trivial and nontrivial gerbes over the toric orbifolds.

2.1. Gale duality and toric Deligne-Mumford stacks. We recall the construction of Gale duality according to [BCS]. Let N be a finitely generated abelian group with rank d . Let

$$\beta : \mathbb{Z}^n \rightarrow N$$

be a map determined by n integral vectors $\{b_1, \dots, b_n\}$ in N . Taking \mathbb{Z}^n and N as \mathbb{Z} -modules, from the homological algebra, there exist projective resolutions \dot{E} and \dot{F} of \mathbb{Z}^n and N satisfying the following diagram

$$\begin{array}{ccc} \dot{E} & \longrightarrow & \mathbb{Z}^n \\ \downarrow & & \downarrow \beta \\ \dot{F} & \longrightarrow & N. \end{array}$$

Let $\text{Cone}(\beta)$ be the mapping cone of the map between \dot{E} and \dot{F} . Then we have an exact sequence of the mapping cone:

$$0 \longrightarrow \dot{F} \longrightarrow \text{Cone}(\beta) \longrightarrow \dot{E}[1] \longrightarrow 0,$$

where $\dot{E}[1]$ is the shifting of \dot{E} by 1. Since \dot{E} is projective as \mathbb{Z} -modules, so we have the exact sequence obtained by applying $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$:

$$0 \longrightarrow \dot{E}[1]^* \longrightarrow \text{Cone}(\beta)^* \longrightarrow \dot{F}^* \longrightarrow 0.$$

Taking cohomology of the above sequence we get the exact sequence

$$(2.1) \quad N^* \xrightarrow{\beta^*} (\mathbb{Z}^n)^* \xrightarrow{\beta^\vee} H^1(\text{Cone}(\beta)^*) \longrightarrow \text{Ext}_{\mathbb{Z}}^1(N, \mathbb{Z}) \longrightarrow 0.$$

Definition 2.1. Let $N^\vee = H^1(\text{Cone}(\beta)^*)$. The map

$$\beta^\vee : (\mathbb{Z}^n)^* \rightarrow N^\vee$$

is called the Gale dual of the map β .

According to [BCS], both N^\vee and β^\vee are well defined up to natural isomorphism.

This construction can be made more clear. Since N has rank d and \mathbb{Z}^n is a free \mathbb{Z} -module, the projection resolutions can be chosen as:

$$\begin{aligned} 0 \longrightarrow \mathbb{Z}^n \longrightarrow 0 &= \dot{E}, \\ 0 \longrightarrow \mathbb{Z}^r \xrightarrow{Q} \mathbb{Z}^{d+r} \longrightarrow 0 &= \dot{F}, \end{aligned}$$

where Q is an integer matrix. Then there is a map $\mathbb{Z}^n \rightarrow \mathbb{Z}^{d+r}$ defined by a matrix B which gives the map between \dot{E} and \dot{F} . The mapping cone $\text{Cone}(\beta)$ is given by the following complex:

$$0 \longrightarrow \mathbb{Z}^{n+r} \xrightarrow{[B, Q]} \mathbb{Z}^{d+r} \longrightarrow 0 = \text{Cone}(\beta).$$

(2.1) is then obtained by applying the snake lemma to the following diagram

$$(2.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & (\mathbb{Z}^{d+r})^* & \longrightarrow & (\mathbb{Z}^{d+r})^* \longrightarrow 0 \\ & & \downarrow & & \downarrow [B, Q]^* & & \downarrow \\ 0 & \longrightarrow & (\mathbb{Z}^n)^* & \longrightarrow & (\mathbb{Z}^{n+r})^* & \longrightarrow & (\mathbb{Z}^r)^* \longrightarrow 0. \end{array}$$

Then $N^\vee = (\mathbb{Z}^{n+r})^* / \text{Im}([B, Q]^*)$ and β^\vee is the composite map of the inclusion $(\mathbb{Z}^n)^* \hookrightarrow (\mathbb{Z}^{n+r})^*$ and the quotient map $(\mathbb{Z}^{n+r})^* \rightarrow (\mathbb{Z}^{n+r})^* / \text{Im}([B, Q]^*)$.

Remark 2.2. If N is free, i.e. there is no torsion part in the group N . Then by (2.2), the Gale dual β^\vee is the quotient map $(\mathbb{Z}^n)^* \rightarrow (\mathbb{Z}^n)^* / \text{Im}([B]^*)$ and we have an exact sequence

$$N^* \xrightarrow{\beta^*} (\mathbb{Z}^n)^* \xrightarrow{\beta^\vee} H^1(\text{Cone}(\beta)^*) \longrightarrow 0.$$

Let $N \rightarrow \overline{N}$ be the natural map of modding out torsion. Then \overline{N} is a lattice. Let Σ be a simplicial fan in the lattice \overline{N} with n rays $\{\rho_1, \dots, \rho_n\}$. Choose n integer vectors $\{b_1, \dots, b_n\}$ such that $\overline{b_i}$ generates the ray ρ_i for $1 \leq i \leq n$. Then we have a map $\beta : \mathbb{Z}^n \rightarrow N$ determined by the vectors $\{b_1, \dots, b_n\}$. We require that β has finite cokernel.

Definition 2.3 ([BCS]). The triple $\Sigma := (N, \Sigma, \beta)$ is called a stacky fan.

We define toric Deligne-Mumford stack from a stacky fan Σ . Since β has finite cokernel, by Proposition 2.2 and 2.3 in [BCS], we have the following exact sequences:

$$(2.3) \quad 0 \longrightarrow (N^\vee)^* \xrightarrow{(\beta^\vee)^*} \mathbb{Z}^n \xrightarrow{\beta} N \longrightarrow \text{Coker}(\beta) \longrightarrow 0,$$

$$(2.4) \quad 0 \longrightarrow N^* \longrightarrow \mathbb{Z}^n \xrightarrow{\beta^\vee} N^\vee \longrightarrow \text{Coker}(\beta^\vee) \longrightarrow 0.$$

Since \mathbb{C}^* is a divisible as a \mathbb{Z} -module, applying $\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to (2.4) gives:

$$(2.5) \quad 1 \longrightarrow \mu \longrightarrow G \xrightarrow{\alpha} (\mathbb{C}^*)^n \longrightarrow T \longrightarrow 1,$$

where $\mu = \text{Hom}_{\mathbb{Z}}(\text{Coker}(\beta^\vee), \mathbb{C}^*)$ is finite, $G = \text{Hom}_{\mathbb{Z}}(N^\vee, \mathbb{Z}^*)$ and T is the d dimensional torus $(\mathbb{C}^*)^d$.

Let $\mathbb{C}[z_1, \dots, z_n]$ be the coordinate ring of the affine variety \mathbb{A}^n . Associated to the simplicial fan Σ , there is an irrelevant ideal J_Σ generated by the elements:

$$(2.6) \quad \left\langle \prod_{\rho_i \notin \sigma} z_i : \sigma \in \Sigma \right\rangle.$$

Let $Z := \mathbb{A}^n \setminus \mathbb{V}(J_\Sigma)$. Then Z is a quasi-affine variety. The torus $(\mathbb{C}^*)^n$ acts on Z naturally since Z is the complement of coordinate subspaces. The algebraic group G acts on the variety Z through the map α in (2.5). Then we have an action groupoid $Z \times G \rightrightarrows Z$.

Definition 2.4 ([BCS]). *The toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ associated to the stacky fan Σ is defined to be the quotient stack $[Z/G]$.*

Since N is a finitely generated abelian group of rank d , we may write

$$N = \mathbb{Z}^d \oplus \mathbb{Z}_{m_1} \oplus \dots \oplus \mathbb{Z}_{m_r}.$$

Then $\overline{N} = \mathbb{Z}^d$, and

$$\overline{\beta} : \mathbb{Z}^n \rightarrow \overline{N}$$

is given by $\{\overline{b}_1, \dots, \overline{b}_n\}$. So $\Sigma_{\text{red}} := (\overline{N}, \Sigma, \overline{\beta})$ is a stacky fan. In the exact sequence (2.5), let $\overline{G} = \text{Im}(\alpha)$, then we have an exact sequence of abelian groups

$$1 \longrightarrow \mu \longrightarrow G \longrightarrow \overline{G} \longrightarrow 1.$$

This is a central extension. By [DP], the quotient stack $[Z/G]$ is the μ -gerbe over the quotient stack $[Z/\overline{G}] =: \mathcal{X}(\Sigma_{\text{red}})$ determined by this central extension.

Remark 2.5. *The stack $\mathcal{X}(\Sigma_{\text{red}})$ can be constructed as follows. Consider the following exact sequences*

$$0 \longrightarrow (\overline{N}^\vee)^\star \longrightarrow \mathbb{Z}^n \xrightarrow{\overline{\beta}} \overline{N} \longrightarrow 0;$$

where $\overline{\beta}$ is given by the vectors $\{\overline{b}_1, \dots, \overline{b}_n\}$, and

$$(2.7) \quad 0 \longrightarrow N^\star \longrightarrow (\mathbb{Z}^n)^\star \xrightarrow{\overline{\beta}^\vee} \overline{N}^\vee \longrightarrow 0.$$

So $A_{d-1}(X(\Sigma)) = \overline{N}^\vee$ and from the construction of Cox [Cox], $\mathcal{X}(\Sigma_{\text{red}}) = [Z/\overline{G}]$, where $\overline{G} = \text{Hom}_{\mathbb{Z}}(\overline{N}^\vee, \mathbb{C}^*)$.

2.2. Construction of toric Deligne-Mumford stacks. It is known that every toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ is a μ -gerbe over the underlying toric orbifold for a finite abelian group μ and some finite abelian gerbes over $\mathcal{X}(\Sigma)$ are again toric Deligne-Mumford stacks, see [Jiang2]. In this section we classify trivial and nontrivial gerbes over toric orbifolds.

Lemma 2.6. *Let \mathbb{Z}^s and \mathbb{Z}^t be two free abelian groups of ranks s and t respectively. Suppose that there is a map $\beta : \mathbb{Z}^s \rightarrow \mathbb{Z}^t$ which is given by an integral $t \times s$ matrix A . Then the dual map $\beta^* : (\mathbb{Z}^t)^* \rightarrow (\mathbb{Z}^s)^*$ is given by the transpose A^t and*

$$\text{coker}(\beta^*) \cong \ker(\beta) \oplus \text{coker} \beta.$$

PROOF. For simplicity, we assume that β has finite cokernel and $s \geq t$. Since the matrix A is an integer matrix, there exist invertible $s \times s$ integer matrix P and an $t \times t$ integer matrix P' such that $P'AP$ is the matrix

$$\begin{bmatrix} a_1 & & \mathbf{0} & \mathbf{0} \\ & \ddots & & \vdots \\ \mathbf{0} & & a_t & \mathbf{0} \end{bmatrix},$$

with $a_1 | \cdots | a_t$. This is the *Smith normal form* (see e.g. [Pra]). It follows that $\text{coker}(\beta) \cong \mathbb{Z}_{a_1} \oplus \cdots \oplus \mathbb{Z}_{a_t}$. After taking dual we get that the map $\beta^* : (\mathbb{Z}^t)^* \rightarrow (\mathbb{Z}^s)^*$ is given by the matrix

$$\begin{bmatrix} a_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & a_t \\ \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}.$$

So it is easy to see that $\text{coker}(\beta^*) = \mathbb{Z}^{s-t} \oplus \mathbb{Z}_{a_1} \oplus \cdots \oplus \mathbb{Z}_{a_t}$. Since the kernel of β is isomorphic to \mathbb{Z}^{s-t} , we complete the proof. \square

Lemma 2.7. *Let Σ be a stacky fan. If the vectors $\{b_1, \dots, b_n\}$ generate the torsion part $\mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r}$ of N , then $r \leq n - d$ and $N^\vee \cong \overline{N}^\vee$.*

PROOF. From the following diagram

$$\begin{array}{ccc} & & \mathbb{Z}^{d+r} \\ & \nearrow & \downarrow \\ \mathbb{Z}^n & \xrightarrow{\beta} & N, \\ & & \downarrow \\ & & 0 \end{array}$$

It is easy to see that if β generate the torision part of N , then $r \leq n - d$.

By the construction of Gale duality in (2.2), $N^\vee \cong (\mathbb{Z}^{n+r})^* / \text{Im}([B, Q]^*)$. Since \overline{N} is free, we have $\overline{N}^\vee \cong (\mathbb{Z}^n)^* / \text{Im}([\overline{B}]^*)$. The map $[B, Q] : \mathbb{Z}^{n+r} \rightarrow \mathbb{Z}^{d+r}$ in the mapping cone $\text{Cone}(\beta)$ has the same cokernel as the map β . Since the map β generate the torsion part of N , the map β has the same cokernel as the map $\overline{\beta}$.

Thus the map $[B, Q]$ has the same cokernel as the map $[\overline{B}] : \mathbb{Z}^n \rightarrow \mathbb{Z}^d$. Also, $\ker[B, Q] \simeq \mathbb{Z}^{n-d} \simeq \ker[\overline{B}]$. By Lemma 2.6, we have

$$N^\vee = \operatorname{coker}[B, Q]^\star \simeq \operatorname{coker}[\overline{B}]^\star = \overline{N}^\vee.$$

□

Let Σ be the stacky fan and Σ_{red} the corresponding reduced stacky fan. Consider the following diagram

$$\begin{array}{ccc} \mathbb{Z}^n & \xrightarrow{\beta} & N \\ id \downarrow & & \downarrow \\ \mathbb{Z}^n & \xrightarrow{\overline{\beta}} & \overline{N}. \end{array}$$

Taking Gale dual yields

$$(2.8) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & \overline{N}^\star & \longrightarrow & \mathbb{Z}^n & \xrightarrow{\overline{\beta}^\vee} & \overline{N}^\vee & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \varphi & & \downarrow & & \\ 0 & \longrightarrow & N^\star & \longrightarrow & \mathbb{Z}^n & \xrightarrow{\beta^\vee} & N^\vee & \longrightarrow & \operatorname{cok}(\beta^\vee) & \longrightarrow & 0. \end{array}$$

Lemma 2.8. *Let $\Sigma = (N, \Sigma, \beta)$ be a stacky fan. If the vectors $\{b_1, \dots, b_n\}$ in the map β generate the torsion part $\mathbb{Z}_{m_1} \oplus \dots \oplus \mathbb{Z}_{m_r}$ of N , then the map φ in (2.8) is diagonalizable over integers.*

PROOF. First in the reduced stacky fan Σ_{red} , the map $\overline{\beta} : \mathbb{Z}^n \rightarrow \mathbb{Z}^d$ is given by $\{\overline{b}_1, \dots, \overline{b}_n\}$. Consider the following diagram

$$(2.9) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}^n & \xrightarrow{id} & \mathbb{Z}^n & \longrightarrow & 0 \\ & & \downarrow \beta' & & \downarrow \beta & & \downarrow \overline{\beta} \\ 0 & \longrightarrow & \mathbb{Z}_{m_1} \oplus \dots \oplus \mathbb{Z}_{m_r} & \longrightarrow & N & \longrightarrow & \overline{N} \longrightarrow 0. \end{array}$$

From the definition of Gale dual in Section 2.1, we have the morphisms of mapping cones $\operatorname{Cone}(\beta')$, $\operatorname{Cone}(\beta)$ and $\operatorname{Cone}(\overline{\beta})$:

$$(2.10) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}^r & \xrightarrow{\overline{Q}} & \mathbb{Z}^r & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}^{n+r} & \xrightarrow{[B, Q]} & \mathbb{Z}^{d+r} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}^n & \xrightarrow{\overline{B}} & \mathbb{Z}^d & \longrightarrow & 0, \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

where \overline{Q} is the diagonal matrix in Q . Dualizing gives the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & (\mathbb{Z}^d)^* & \xrightarrow{\overline{B}^*} & (\mathbb{Z}^n)^* & \xrightarrow{\overline{\beta}^\vee} & \overline{N}^\vee & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow i & & \downarrow \varphi & & \\
 (2.11) \quad 0 & \longrightarrow & (\mathbb{Z}^{d+r})^* & \xrightarrow{[B,Q]^*} & (\mathbb{Z}^{n+r})^* & \xrightarrow{\pi} & N^\vee & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & (\mathbb{Z}^r)^* & \xrightarrow{\overline{Q}^*} & (\mathbb{Z}^r)^* & \longrightarrow & \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r} & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
 \end{array}$$

where $\pi \circ i = \beta^\vee$. Since the map φ is induced from the map i in (2.11), it is given by an integer matrix A . Then from the general fact in the finitely generated group theory there exist integer matrices P, P' such that PAP' is a diagonal matrix with entries $n_1, \dots, n_s, 0, \dots, 0$ which satisfy the condition $n_1 | \cdots | n_s$. This is again the Smith normal form. From the diagram (2.11) the third column is exact and the cokernel is $\mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r}$, so the diagonal matrix given by φ is of the form

$$\begin{bmatrix}
 1 & & & & & \\
 & \ddots & & & & \\
 & & 1 & & & \\
 & & & m_1 & & \\
 & & & & \ddots & \\
 & & & & & m_r
 \end{bmatrix}.$$

□

Proposition 2.9. *Let $\Sigma = (N, \Sigma, \beta)$ be a stacky fan and $\mu = \mu_{m_1} \times \cdots \times \mu_{m_r}$. If the vectors $\{b_1, \dots, b_n\}$ in the map β generate the torsion part $\mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r}$ of N , then $\mathcal{X}(\Sigma)$ is a nontrivial μ -gerbe over the toric orbifold $\mathcal{X}(\Sigma_{\text{red}})$.*

PROOF. Applying $\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to the diagram (2.8) yields

$$\begin{array}{ccccccccc}
 1 & \longrightarrow & \mu & \longrightarrow & G & \xrightarrow{\alpha} & (\mathbb{C}^*)^n & \longrightarrow & T \longrightarrow 1 \\
 & & \downarrow & & \downarrow \alpha(\varphi) & & \downarrow & & \downarrow \\
 (2.12) \quad 1 & \longrightarrow & 1 & \longrightarrow & \overline{G} & \xrightarrow{\overline{\alpha}} & (\mathbb{C}^*)^n & \longrightarrow & T \longrightarrow 1,
 \end{array}$$

where the map $\alpha(\varphi)$ is given by the diagonal matrix:

$$(2.13) \quad \begin{bmatrix} (\cdot)^1 & & & & \\ & \ddots & & & \\ & & (\cdot)^1 & & \\ & & & (\cdot)^{m_1} & \\ & & & & \ddots \\ & & & & & (\cdot)^{m_r} \end{bmatrix},$$

since by Lemma 2.8 the map φ in (2.8) is diagonalizable. By Lemma 2.7, $N^\vee \cong \overline{N}^\vee$ and so $G \cong \overline{G}$. So we get an exact sequence

$$(2.14) \quad 1 \longrightarrow \mu \xrightarrow{\varphi} G \longrightarrow \overline{G} \longrightarrow 1$$

which is a central extension, where $\mu = \text{Hom}_{\mathbb{Z}}(\text{cok}(\beta^\vee), \mathbb{C}^*) = \mu_{m_1} \times \cdots \times \mu_{m_r}$. We can decompose the left side of the diagram (2.12) according to (2.13) to get the following diagram for each μ_{m_i} :

$$(2.15) \quad \begin{array}{ccccccccc} 1 & \longrightarrow & \mu_{m_i} & \longrightarrow & \mathbb{C}^* & \xrightarrow{\alpha_i} & (\mathbb{C}^*)^n & \longrightarrow & T & \longrightarrow & 1 \\ & & \downarrow & & \downarrow (\cdot)^{m_i} & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & 1 & \longrightarrow & \mathbb{C}^* & \xrightarrow{\overline{\alpha}_i} & (\mathbb{C}^*)^n & \longrightarrow & T & \longrightarrow & 1. \end{array}$$

The corresponding component $\mathbb{C}^* \subset G$ determines a line bundle M_i over the toric orbifold $\mathcal{X}(\Sigma_{\text{red}})$. The central extension (2.14) is nontrivial, because $\overline{G} \cong G$ and the map φ in (2.14) is nontrivial on each component. So from the definition of gerbes, the quotient stack $\mathcal{X}(\Sigma) = [Z/G]$ is a nontrivial μ -gerbe over the toric orbifold $\mathcal{X}(\Sigma_{\text{red}}) = [Z/\overline{G}]$ coming from the direct sum of line bundles $\oplus_i M_i$. \square

Remark 2.10. Recall that the stack of roots of a line bundle can be constructed as follows. More details can be found in Appendix B of [AGV2], or [Ca]. Let L be a line bundle over a variety (or an Artin stack) X . Let m be a positive integer and consider the Kummer exact sequence:

$$1 \longrightarrow \mu_m \longrightarrow \mathbb{C}^* \xrightarrow{(\cdot)^m} \mathbb{C}^* \longrightarrow 1.$$

Then we have the quotient stack $[\mathbb{C}^*/\mathbb{C}^*] \simeq B\mu_m$, where the action is given by $\lambda \cdot x = \lambda^m x$. Let L^* be the complement of the zero section in the total space of L . The following twist

$${}^m\sqrt{L} := [L^*/\mathbb{C}^*] = [L^* \times_{\mathbb{C}^*} \mathbb{C}^*/\mathbb{C}^*],$$

is the μ_m -gerbe over X coming from the line bundle L , which is called the stack of m -th root of L . ${}^m\sqrt{L}$ may be viewed as a toric stack bundle in the sense of [Jiang1] since the stack $[\mathbb{C}^*/\mathbb{C}^*]$ is a toric Deligne-Mumford stack.

In the proof of Proposition 2.9, the toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ is obtained by applying a sequence of root constructions to line bundles M_i over the toric orbifold $\mathcal{X}(\Sigma_{\text{red}})$, i.e.

$$\mathcal{X}(\Sigma) \cong {}^{m_1}\sqrt{M_1} \times_{\mathcal{X}(\Sigma_{\text{red}})} \cdots \times_{\mathcal{X}(\Sigma_{\text{red}})} {}^{m_r}\sqrt{M_r}.$$

Similar descriptions have also been obtained independently by F. Perroni [Perroni].

Example 2.11. Let $N = \mathbb{Z}^2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$, and $\beta : \mathbb{Z}^4 \rightarrow N$ be given by the vectors

$$\{b_1 = (1, 0, 1, 0), b_2 = (0, 1, 0, 0), b_3 = (-1, 2, 0, 0), b_4 = (0, -1, 0, 1)\}.$$

Then $\overline{N} = \mathbb{Z}^2$ and let Σ be the complete fan in \mathbb{R}^2 generated by $\{\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4\}$. The fan Σ is the fan of Hirzebruch surface \mathbb{F}_2 . Then $\Sigma = (N, \Sigma, \beta)$ is a stacky fan. We compute that $\beta^\vee : \mathbb{Z}^4 \rightarrow N^\vee = \mathbb{Z}^2$ is given by the matrix

$$\begin{bmatrix} 2 & -4 & 2 & 0 \\ 0 & 4 & 0 & 4 \end{bmatrix}.$$

The reduced stacky fan is $\Sigma_{\text{red}} = (\mathbb{Z}^2, \Sigma, \bar{\beta})$, where $\bar{\beta} : \mathbb{Z}^4 \rightarrow \mathbb{Z}^2$ is given by $\{\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4\}$. The Gale dual $\bar{\beta}^\vee : \mathbb{Z}^4 \rightarrow N^\vee = \mathbb{Z}^2$ is given by the matrix

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

So in this example the diagram (2.8) is:

$$(2.16) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^4 & \xrightarrow{\beta^\vee} & \mathbb{Z}^2 & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \text{id} & & \downarrow \varphi & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^4 & \xrightarrow{\beta^\vee} & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}_2 \oplus \mathbb{Z}_4 & \longrightarrow & 0, \end{array}$$

where φ is the diagonal matrix

$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

So from (2.16) we get the following exact sequence:

$$(2.17) \quad \begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 \times \mu_4 & \longrightarrow & (\mathbb{C}^*)^2 & \longrightarrow & (\mathbb{C}^*)^4 & \longrightarrow & (\mathbb{C}^*)^2 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \alpha(\varphi) & & \downarrow \text{id} & & \downarrow & & \\ 1 & \longrightarrow & 1 & \longrightarrow & (\mathbb{C}^*)^2 & \longrightarrow & (\mathbb{C}^*)^4 & \longrightarrow & (\mathbb{C}^*)^2 & \longrightarrow & 1. \end{array}$$

The toric Deligne-Mumford stack $\mathcal{X}(\Sigma) = [\mathbb{C}^4 - \mathbb{V}(J_\Sigma)/(\mathbb{C}^*)^2]$, where the action is given by the transpose of the matrix β^\vee . Let L_1, L_2 be the two line bundles over \mathbb{F}_2 which are the two canonical generators in the Picard group \mathbb{Z}^2 of \mathbb{F}_2 . The toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ is a $\mu_2 \times \mu_4$ -gerbe over the Hirzebruch surface $\mathcal{X}(\Sigma_{\text{red}}) = \mathbb{F}_2$ coming from the direct sum of line bundles $L_1 \oplus L_2$. $\mathcal{X}(\Sigma)$ can be constructed by taking square root and quartic root of line bundles L_1 and L_2 respectively: $\mathcal{X}(\Sigma) \cong \sqrt[2]{L_1} \times_{\mathbb{F}_2} \sqrt[4]{L_2}$.

Remark 2.12. In the map $\beta : \mathbb{Z}^n \rightarrow N$, if the components of the torsion part of b_i are zero, then it is easy to check that $N^\vee \cong \overline{N}^\vee \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r}$ and $\mathcal{X}(\Sigma) = \mathcal{X}(\Sigma_{\text{red}}) \times \mathcal{B}\mu$ with $\mu = \mu_{m_1} \times \cdots \times \mu_{m_r}$.

Proposition 2.13. Every toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ has a decomposition:

$$\mathcal{X}(\Sigma) \cong \mathcal{X}(\Sigma') \times \mathcal{B}\mu,$$

where $\mathcal{X}(\Sigma')$ is a nontrivial gerbe over the toric orbifold $\mathcal{X}(\Sigma'_{\text{red}})$ and $\mathcal{B}\mu$ is the classifying stack for a finite abelian group μ .

PROOF. Consider the map $\beta : \mathbb{Z}^n \rightarrow N$ given by integral vectors $\{b_1, \dots, b_n\} \subset N$. Let $N = \mathbb{Z}^d \oplus \mathbb{Z}_{m_1} \oplus \dots \oplus \mathbb{Z}_{m_r}$ and $N_{\text{tor}} = \mathbb{Z}_{m_1} \oplus \dots \oplus \mathbb{Z}_{m_r}$. Then there exists a subgroup $N'_{\text{tor}} \subset N_{\text{tor}}$ such that $\{b_1, \dots, b_n\}$ generate N'_{tor} . Let $N' = \mathbb{Z}^d \oplus N'_{\text{tor}}$ and let $\beta' : \mathbb{Z}^n \rightarrow N'$ be the map given by $\{b_1, \dots, b_n\}$. Then $\Sigma' = (N', \Sigma, \beta')$ is a stacky fan such that $\{b'_1, \dots, b'_n\}$ generate the torsion part of N' .

Let Σ_{red} be the corresponding reduced stacky fan. Then by Lemma 2.7, in the Gale dual $(\beta')^\vee : \mathbb{Z}^n \rightarrow (N')^\vee$ and $\bar{\beta}^\vee : \mathbb{Z}^n \rightarrow (\bar{N})^\vee$, $(N')^\vee \cong (\bar{N})^\vee$. Since the cokernel of the map β is

$$\text{cok}(\bar{\beta}) \oplus N_{\text{tor}}/N'_{\text{tor}} \cong \text{cok}(\beta') \oplus N_{\text{tor}}/N'_{\text{tor}},$$

by Lemma 2.6, the Gale dual map of β is

$$\beta^\vee : \mathbb{Z}^n \rightarrow (N')^\vee \oplus N_{\text{tor}}/N'_{\text{tor}}.$$

Let $\mu = N_{\text{tor}}/N'_{\text{tor}}$. By Lemma 2.9 and Remark 2.12 the proposition follows. \square

3. LINE BUNDLES OVER TORIC DELIGNE-MUMFORD STACKS

In this section we prove that the Picard group of a toric Deligne-Mumford stacks is isomorphic to N^\vee in the Gale dual $\beta^\vee : \mathbb{Z}^n \rightarrow N^\vee$ of $\beta : \mathbb{Z}^n \rightarrow N$.

Let $\mathcal{X}(\Sigma) = [Z/G]$ be a toric Deligne-Mumford stack associated to the stacky fan $\Sigma = (N, \Sigma, \beta)$. In the case of a quotient stack, a line bundle \mathcal{L} on $\mathcal{X}(\Sigma)$ is a G -equivariant line bundle L on Z . A G -equivariant line bundle on Z is a line bundle L on Z together with an isomorphism $\varphi : pr^*L \rightarrow \mu^*L$, where $pr : G \times Z \rightarrow Z$ is the projection and $\mu : G \times Z \rightarrow Z$ is the action. Let $\text{Pic}(\mathcal{X}(\Sigma))$ denote the Picard group of $\mathcal{X}(\Sigma)$.

Proposition 3.1. $\text{Pic}(\mathcal{X}(\Sigma)) \cong N^\vee$.

PROOF. Since $G = \text{Hom}_{\mathbb{Z}}(N^\vee, \mathbb{C}^*)$, Pontryagin duality implies that the character group G^\vee of G is isomorphic to N^\vee .

As discussed above, $\mathcal{X}(\Sigma) = [Z/G]$ where $Z = \mathbb{A}^n \setminus \mathbb{V}(J_\Sigma)$ with $\text{codim } \mathbb{V}(J_\Sigma) \geq 2$. This implies that $\text{Pic}(Z) \simeq \text{Pic}(\mathbb{A}^n) \simeq \mathbb{Z}$, generated by the trivial line bundle.

Let $\mathcal{L} \in \text{Pic}(\mathcal{X}(\Sigma))$, then \mathcal{L} corresponds to a G -equivariant line bundle $L \rightarrow Z$, which is trivial but not necessarily G -equivariantly trivial. The G -action on L is given by a character $\chi_{\mathcal{L}} : G \rightarrow \mathbb{C}^*$. Thus there is a map

$$\text{Pic}(\mathcal{X}(\Sigma)) \rightarrow G^\vee, \quad \mathcal{L} \rightarrow \chi_{\mathcal{L}}.$$

On the other hand, any character $\chi : G \rightarrow \mathbb{C}^*$ gives a G -equivariant line bundle L_χ over Z , hence a line bundle on $\mathcal{X}(\Sigma)$. The result follows. \square

From the results in [EG], [Kr] that the Picard group of a quotient stack is isomorphic to its first Chow group, we have:

Corollary 3.2. *There is an isomorphism $A^1(\mathcal{X}(\Sigma), \mathbb{Z}) \cong N^\vee$.*

For each ray ρ_i , we define a line bundle \mathcal{L}_i over $\mathcal{X}(\Sigma)$ to be the trivial line bundle \mathbb{C} with the G action on \mathbb{C} given by the i -th component of the map $\alpha : G \rightarrow (\mathbb{C}^*)^n$. We can similarly define a line bundle L_i over the toric orbifold $\mathcal{X}(\Sigma_{\text{red}})$. This line bundle L_i is the trivial line bundle \mathbb{C} over Z on which the action of \overline{G} is through the i -th component of the map $\overline{\alpha} : \overline{G} \rightarrow (\mathbb{C}^*)^n$, where $\overline{\alpha}$ is obtained by taking $\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to the map $\overline{\beta}^\vee$ in (2.7).

For the stacky fan Σ and its reduced stacky fan Σ_{red} , we consider again the diagram (2.8). Since we have that $\text{cok}(\beta^\vee) = \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_{n-d}}$ by Lemma 2.8, the map φ is given by the diagonal matrix

$$\begin{bmatrix} m_1 & & \\ & \ddots & \\ & & m_{n-d} \end{bmatrix},$$

where some m_i 's are 1 since $r \leq n - d$.

Let $\mathbf{t} := \{t_1, \dots, t_{n-d}\}$ be the generators of \overline{N}^\vee so that the map φ is diagonalized. Let $\{e_1, \dots, e_n\}$ be the standard basis of \mathbb{Z}^n . Let $\mathbf{x} = (x_i)$ and $\mathbf{t} = (t_i)$ be column vectors, then there exist a matrix A such that $\mathbf{x} = A\mathbf{t}$. Then under these bases the map $\overline{\beta}^\vee$ is given by the matrix A^t . Suppose that the map β^\vee is given by a matrix B , then we have $B = MA^t$. Since $\overline{\beta}^\vee$ is surjective, there exists an integral matrix C such that $\mathbf{t} = C\mathbf{x}$, where $\mathbf{x} := (x_1, \dots, x_n)^t$ with $x_j := Ae_j$. Let $\tilde{\mathbf{x}} := (\tilde{x}_1, \dots, \tilde{x}_n)^t$ be defined by

$$(3.1) \quad \tilde{\mathbf{x}} = AM\mathbf{t} = AMC\mathbf{x}.$$

Then every \tilde{x}_i is an integral linear combination of x_i 's by the above formula. We call the formula (3.1) the associated formula of the stacky fan Σ .

Let $\pi : \mathcal{X}(\Sigma) \rightarrow \mathcal{X}(\Sigma_{\text{red}})$ be the rigidification map. For the line bundle L_j over $\mathcal{X}(\Sigma_{\text{red}})$ corresponding to ray ρ_j , we have the pullback $\pi^*(L_j) = \mathcal{L}_j$ over $\mathcal{X}(\Sigma)$. From Lemma 2.8, we have the morphism $\varphi : \overline{N}^\vee \rightarrow N^\vee$ after choosing the basis of $\overline{N}^\vee \cong N^\vee$ which is diagonalizable. Let $id : \overline{N}^\vee \rightarrow N^\vee$ be the identity morphism under the basis, dualizing we an isomorphism

$$id : G \xrightarrow{\cong} \overline{G}.$$

Under this isomorphism, we can take L_j as a line bundle over $\mathcal{X}(\Sigma)$ using the character $\overline{\alpha}_j : \overline{G} \rightarrow \mathbb{C}$ in the j -th component of the map $\overline{\alpha}$. Suppose that in the formula (3.1),

$$\tilde{x}_i = \sum_{j=1}^n a_{j,i} x_j,$$

where $a_{1,i}, \dots, a_{n,i}$ are integers, then we have the following proposition.

Proposition 3.3. $\mathcal{L}_i = \bigotimes_j L_j^{\otimes a_{j,i}}.$

PROOF. From the construction of the line bundle \mathcal{L}_i on the toric Deligne-Mumford stack the proposition can be easily proved from the diagram (2.8) and the formula (3.1). \square

4. INTEGRAL CHOW RING OF TORIC DELIGNE-MUMFORD STACKS

In this section we study the integral Chow ring of toric Deligne-Mumford stacks. For references of integral Chow ring of stacks, see [EG] and [Kr]. In this section, every Deligne-Mumford stack $\mathcal{X}(\Sigma)$ is semi-projective satisfying the condition of Lemma 2.9. We use the results in [Iwa1], [Iwa2] concerning the integral Chow ring of a simplicial toric orbifold.

4.1. The Chow ring of stack of roots of a line bundle. In this section we compute the Chow ring with integer coefficients of the stack $\sqrt[m]{L}$ of m -th root of a line bundle L .

Let \mathbb{C}^* act on the total space of L by acting trivially on the base X and acting by $\lambda \cdot x := \lambda^m x$ on the fiber. As recalled in Remark 2.10, $\sqrt[m]{L}$ is the stack quotient $[L^*/\mathbb{C}^*]$ with respect to this \mathbb{C}^* -action.

Let $i : X \hookrightarrow L$ be the inclusion of the zero section, and $j : L \setminus X \hookrightarrow L$ the inclusion of its complement. Then we have an exact sequence on the \mathbb{C}^* -equivariant Chow groups:

$$(4.1) \quad A_*(X)_{\mathbb{C}^*} \xrightarrow{i_*} A_*(L)_{\mathbb{C}^*} \xrightarrow{j^*} A_*(L \setminus X)_{\mathbb{C}^*} \longrightarrow 0,$$

where i_* is the pushforward and j^* is the flat pullback. By [EG],

$$A^*([L^*/\mathbb{C}^*]) = A_{\mathbb{C}^*}^*(L \setminus X).$$

Let $\pi : L \rightarrow X$ be the structure map of the line bundle. Then we have

$$\pi^* : A_*(X)_{\mathbb{C}^*} \xrightarrow{\cong} A_{*+1}(L)_{\mathbb{C}^*}.$$

Let

$$(4.2) \quad b := (\pi^*)^{-1} \circ i_* : A_*(X)_{\mathbb{C}^*} \rightarrow A_{*-1}(X)_{\mathbb{C}^*}.$$

We may rewrite the sequence (4.1) as

$$(4.3) \quad A_*(X)_{\mathbb{C}^*} \xrightarrow{b} A_{*-1}(X)_{\mathbb{C}^*} \xrightarrow{j^*} A_*(L \setminus X)_{\mathbb{C}^*} \longrightarrow 0.$$

This implies that

$$(4.4) \quad A_*([L^*/\mathbb{C}^*]) \cong A_{*-1}(X)_{\mathbb{C}^*}/\text{Im}(b).$$

Clearly the map b is given by the \mathbb{C}^* -equivariant first Chern class of the line bundle L , which is $c_1(L) - mt$, where $c_1(L)$ is the non-equivariant first Chern class of L , and t is the equivariant parameter. Thus we have the following proposition:

Proposition 4.1. *The Chow ring $A^*([L^*/\mathbb{C}^*])$ is isomorphic to the quotient ring*

$$\frac{A^*(X)[t]}{(c_1(L) - mt)}.$$

PROOF. Since \mathbb{C}^* acts trivially on X , we have $A^*(X)_{\mathbb{C}^*} = A^*(X)[t]$. Since $c_1(L) - mt$ is not a zero divisor, (4.3) is exact on the left, and the image of b is the ideal generated by $c_1(L) - mt$. \square

4.2. Integral Chow ring of toric Deligne-Mumford stacks. Let Σ be a stacky fan such that the map β generate the torsion part of the group N and $\mathcal{X}(\Sigma)$ the associated toric Deligne-Mumford stack. Let $\mathcal{X}_{orb}(\Sigma)$ be the underlying toric orbifold given by the simplicial fan Σ . Let

$$(4.5) \quad \frac{\mathbb{Z}[x_i : \rho_i \in \Sigma(1)]}{(I_\Sigma + Cir(\Sigma))}$$

be the Stanley-Reisner ring of the fan Σ , where x_i corresponds to the torus invariant divisor D_{ρ_i} , $Cir(\Sigma)$ is the ideal generated by the linear relations:

$$\left(\sum_{\rho_i \in \Sigma} \theta(v_i) x_i \right)_{\theta \in N^*},$$

where v_i is the first lattice point of the ray ρ_i , and I_Σ is the ideal generated by square-free monomials

$$\{x_{i_1} \cdots x_{i_k} : \rho_{i_1} + \cdots + \rho_{i_k} \text{ is not a cone } \sigma \in \Sigma\}$$

in (1.2).

Proposition 4.2 ([Iwa1]). *The integral Chow ring $A^*(\mathcal{X}_{orb}(\Sigma), \mathbb{Z})$ is isomorphic to (4.5).*

Consider $\Sigma_{\text{red}} = (\overline{N}, \Sigma, \overline{\beta})$, where $\overline{\beta} : \mathbb{Z}^n \rightarrow \overline{N}$ is given by the vectors $\{\overline{b}_1, \dots, \overline{b}_n\}$, and the toric Deligne-Mumford stack $\mathcal{X}(\Sigma_{\text{red}}) = [Z/\overline{G}]$. Let $C(\Sigma_{\text{red}})$ is the ideal generated by the linear relations:

$$\left(\sum_{\rho_i \in \Sigma} \theta(b_i) x_i \right)_{\theta \in N^*}$$

in (1.3).

Proposition 4.3 ([Iwa2]). *There is an isomorphism of rings:*

$$A^*(\mathcal{X}(\Sigma_{\text{red}}), \mathbb{Z}) \cong \frac{\mathbb{Z}[x_i : \rho_i \in \Sigma(1)]}{(I_\Sigma + C(\Sigma_{\text{red}}))}.$$

Remark 4.4. *Let $\Sigma = (N, \Sigma, \beta)$ be a stacky fan. For the simplicial fan Σ , the toric orbifold $\mathcal{X}_{orb}(\Sigma)$ associated to Σ has stack structures in codimension at least 2. The toric orbifold $\mathcal{X}(\Sigma_{\text{red}})$ has stack structures in codimension at least 1. The toric orbifold $\mathcal{X}(\Sigma_{\text{red}})$ can be obtained from $\mathcal{X}_{orb}(\Sigma)$ by taking roots of divisors. Since we don't need this result here, we omit the details.*

In view of Proposition 2.9, the idea of proving Theorem 1.1 is to compute the integral Chow ring of $\mathcal{X}(\Sigma)$ by combining Propositions 4.3 and 4.1.

Proof of Theorem 1.1. As in the proof of Proposition 2.9, let $M_i \rightarrow \mathcal{X}(\Sigma_{\text{red}})$ be the line bundle over the toric orbifold given by one generator in \overline{N}^\vee and Let m_i be the corresponding positive integer. The toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ is a nontrivial $\mu_{m_1} \times \cdots \times \mu_{m_r}$ -gerbe over the toric orbifold $\mathcal{X}(\Sigma_{\text{red}})$ obtained

from a sequence of root gerbe constructions determined by the line bundles M_i for $1 \leq i \leq n - d$.

Since the integral Chow ring $A^*(\mathcal{X}(\Sigma_{\text{red}}), \mathbb{Z})$ is generated by the Picard group \overline{N}^\vee and $\frac{\mathbb{Z}[x_i : \rho_i \in \Sigma(1)]}{C(\Sigma_{\text{red}})} \cong \overline{N}^\vee$, so by Proposition 4.3,

$$A^*(\mathcal{X}(\Sigma_{\text{red}}); \mathbb{Z}) \cong \frac{\mathbb{Z}[x_i : \rho_i \in \Sigma(1)]}{(\mathcal{I}_{\Sigma_{\text{red}}} + C(\Sigma_{\text{red}}))} \cong \frac{\mathbb{Z}[t_1, \dots, t_{n-d}]}{I_{\Sigma_{\text{red}}}},$$

where $\{t_1, \dots, t_{n-d}\}$ is a basis of N^\vee , the ideal $I_{\Sigma_{\text{red}}}$ is obtained from $\mathcal{I}_{\Sigma_{\text{red}}}$ by expressing x_i 's in terms of t_i 's. By Proposition 4.1, the Chow ring $A^*(\mathcal{X}(\Sigma), \mathbb{Z})$ is isomorphic to the ring obtained from $\frac{\mathbb{Z}[t_1, \dots, t_{n-d}]}{I_{\Sigma_{\text{red}}}}$ by replacing the canonical generators $\{t_1, \dots, t_{n-d}\}$ by $\{m_1 t_1, \dots, m_{n-d} t_{n-d}\}$. In view of (3.1), this is isomorphic to the Stanley-Reisner ring $SR(\Sigma)$ of the stacky fan Σ since in the ring $SR(\Sigma)$, the ideal \mathcal{I}_Σ is obtained from the ideal $\mathcal{I}_{\Sigma_{\text{red}}}$ in (1.5) replacing x_i by \tilde{x}_i for each ray ρ_i . \square

By Proposition 2.13, every toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ has a decomposition: $\mathcal{X}(\Sigma) \cong \mathcal{X}(\Sigma') \times \mathcal{B}\mu$, where $\mathcal{X}(\Sigma')$ is a nontrivial gerbe over the toric orbifold $\mathcal{X}(\Sigma_{\text{red}})$ and $\mathcal{B}\mu$ is the classifying stack of a finite abelian group $\mu = \mu_{r_1} \times \dots \times \mu_{r_s}$. It is known that

$$A^*(\mathcal{B}\mu, \mathbb{Z}) \cong \mathbb{Z}[t_1, \dots, t_s] / (r_1 t_1, \dots, r_s t_s).$$

Thus we have

Proposition 4.5. *The integral Chow ring of $\mathcal{X}(\Sigma)$ is given by*

$$A^*(\mathcal{X}(\Sigma), \mathbb{Z}) \cong A^*(\mathcal{X}(\Sigma'), \mathbb{Z})[t_1, \dots, t_s] / (r_1 t_1, \dots, r_s t_s).$$

5. THE INTEGRAL ORBIFOLD CHOW RING

In this section we compute the integral orbifold Chow ring of toric Deligne-Mumford stacks.

5.1. The inertia stack. Let $\mathcal{X}(\Sigma)$ be a toric Deligne-Mumford stack associated to the stacky fan $\Sigma = (N, \Sigma, \beta)$. For a cone σ in the simplicial fan Σ , let $\text{link}(\sigma) = \{b_i : \rho_i + \sigma \text{ is a cone in } \Sigma\}$. Then we have a quotient stacky fan $\Sigma/\sigma = (N(\sigma), \Sigma/\sigma, \beta(\sigma))$, where

$$\beta(\sigma) : \mathbb{Z}^l \rightarrow N(\sigma)$$

is given by the images of $\{b_i\}$'s in $\text{link}(\sigma)$. Let $m := |\sigma|$, then $\dim(N_\sigma) = m$ since σ is simplicial. Consider the commutative diagrams

$$(5.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}^{l+m} & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \mathbb{Z}^{n-l-m} \longrightarrow 0 \\ & & \downarrow \tilde{\beta} & & \downarrow \beta & & \downarrow \\ 0 & \longrightarrow & N & \xrightarrow{\cong} & N & \longrightarrow & 0 \longrightarrow 0, \end{array}$$

and

$$(5.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}^m & \longrightarrow & \mathbb{Z}^{l+m} & \longrightarrow & \mathbb{Z}^l \longrightarrow 0 \\ & & \downarrow \beta_\sigma & & \downarrow \tilde{\beta} & & \downarrow \beta(\sigma) \\ 0 & \longrightarrow & N_\sigma & \longrightarrow & N & \longrightarrow & N(\sigma) \longrightarrow 0. \end{array}$$

Applying the Gale dual yields

$$(5.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}^{n-l-m} & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \mathbb{Z}^{l+m} \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow \beta^\vee & & \downarrow \tilde{\beta}^\vee \\ 0 & \longrightarrow & \mathbb{Z}^{n-l-m} & \longrightarrow & N^\vee & \xrightarrow{\phi_1} & \tilde{N}^\vee \longrightarrow 0, \end{array}$$

and

$$(5.4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}^l & \longrightarrow & \mathbb{Z}^{l+m} & \longrightarrow & \mathbb{Z}^m \longrightarrow 0 \\ & & \downarrow \beta(\sigma)^\vee & & \downarrow \tilde{\beta}^\vee & & \downarrow \beta_\sigma^\vee \\ 0 & \longrightarrow & N(\sigma)^\vee & \xrightarrow{\phi_2} & \tilde{N}^\vee & \longrightarrow & N_\sigma^\vee \longrightarrow 0. \end{array}$$

Since $\mathbb{Z}^m \cong N_\sigma$, we have $N_\sigma^\vee = 0$. Applying $\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to (5.3), (5.4) yields

$$(5.5) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \tilde{G} & \xrightarrow{\varphi_1} & G & \longrightarrow & (\mathbb{C}^*)^{n-l-m} \longrightarrow 1 \\ & & \downarrow \tilde{\alpha} & & \downarrow \alpha & & \downarrow \cong \\ 1 & \longrightarrow & (\mathbb{C}^*)^{l+m} & \longrightarrow & (\mathbb{C}^*)^n & \longrightarrow & (\mathbb{C}^*)^{n-l-m} \longrightarrow 1, \end{array}$$

and

$$(5.6) \quad \begin{array}{ccccccc} 1 & \longrightarrow & 1 & \longrightarrow & \tilde{G} & \xrightarrow{\cong} & G(\sigma) \longrightarrow 1 \\ & & \downarrow & & \downarrow \tilde{\alpha} & & \downarrow \alpha(\sigma) \\ 1 & \longrightarrow & (\mathbb{C}^*)^m & \longrightarrow & (\mathbb{C}^*)^{l+m} & \longrightarrow & (\mathbb{C}^*)^l \longrightarrow 1. \end{array}$$

Let $Z(\sigma) = \mathbb{A}^l \setminus \mathbb{V}(J_{\Sigma/\sigma})$, where $J_{\Sigma/\sigma}$ is the irrelevant ideal of the quotient simplicial fan Σ/σ . By the definition of toric Deligne-Mumford stack, we have $\mathcal{X}(\Sigma/\sigma) = [Z(\sigma)/G(\sigma)]$, where the action of $G(\sigma)$ is via the map $\alpha(\sigma)$ in (5.6).

Proposition 5.1. *If σ is a cone in the simplicial fan Σ , then $\mathcal{X}(\Sigma/\sigma)$ is a closed substack of $\mathcal{X}(\Sigma)$.*

PROOF. Let $W(\sigma)$ be the subvariety of Z defined by the ideal $J(\sigma) := \langle z_i : \rho_i \subseteq \sigma \rangle$. Then $W(\sigma)$ contains the \mathbb{C} -points $z \in \mathbb{C}^n$ such that the cone spanned by $\{\rho_i : z_i = 0\}$ containing σ belongs to Σ . Then the \mathbb{C} -point z in $W(\sigma)$ such that $\rho_i \not\subseteq \sigma \cup \text{link}(\sigma)$ implies that $z_i \neq 0$. This implies that $W(\sigma) \cong Z(\sigma) \times (\mathbb{C}^*)^{n-l-m}$. It is clear that $W(\sigma)$ is invariant under the G -action.

Let $\varphi_0 : Z(\sigma) \rightarrow W(\sigma)$ be the inclusion given by $z \mapsto (z, 1)$. Then we have a morphism of groupoids $\varphi_0 \times \varphi_1 : Z(\sigma) \times G(\sigma) \rightrightarrows W(\sigma) \times G$ which induces a morphism of stacks $\varphi : [Z(\sigma)/G(\sigma)] \rightarrow [W(\sigma)/G]$. To prove that it is an

isomorphism, we first prove that the following diagram is cartesian:

$$\begin{array}{ccc} Z(\sigma) \times G(\sigma) & \xrightarrow{\varphi_0 \times \varphi_1} & W(\sigma) \times G \\ (s,t) \downarrow & & \downarrow (s,t) \\ Z(\sigma) \times Z(\sigma) & \xrightarrow{\varphi_0 \times \varphi_0} & W(\sigma) \times W(\sigma). \end{array}$$

This is easy to prove. Given an element $(z_1, z_2) \in Z(\sigma) \times Z(\sigma)$, under the map $\varphi_0 \times \varphi_0$, we get $((z_1, 1), (z_2, 1)) \in W(\sigma) \times W(\sigma)$. If there is an element $g \in G$ such that $g(z_1, 1) = (z_2, 1)$, then from the exact sequence in the first row of (5.5), there is an element $g(\sigma) \in G(\sigma)$ such that $g(\sigma)z_1 = z_2$. Thus we have an element $(z_1, g(\sigma)) \in Z(\sigma) \times G(\sigma)$. So the morphism $\varphi : [Z(\sigma)/G(\sigma)] \rightarrow [W(\sigma)/G]$ is injective. Let $(z, 1)$ be an element in $W(\sigma)$, then there exists an element $g \in (\mathbb{C}^*)^{n-l-m}$ such that $g(z, 1) = (z, 1)$. By (5.5), g determines an element in G , so φ is surjective and φ is an isomorphism. Clearly the stack $[W(\sigma)/G]$ is a closed substack of $\mathcal{X}(\Sigma)$, so $\mathcal{X}(\Sigma/\sigma) = [Z(\sigma)/G(\sigma)]$ is also a closed substack of $\mathcal{X}(\Sigma)$. \square

Remark 5.2. *Proposition 5.1 is Proposition 4.2 of [BCS]. However the proof given there has a gap: some incorrect exact sequences were used. We choose to give a new proof here. One can prove this result using extended stacky fan defined in [Jiang1] since the quotient stacky fan is naturally an extended stacky fan.*

Following [BCS], for each top dimensional cone σ in Σ , denote by $\text{Box}(\sigma)$ the set of elements $v \in N$ such that $\bar{v} = \sum_{\rho_i \subseteq \sigma} a_i \bar{b}_i$ for some $0 \leq a_i < 1$. The elements in $\text{Box}(\sigma)$ are in one-to-one correspondence with the elements in the finite group $N(\sigma) = N/N_\sigma$, where $N(\sigma)$ is a local isotropy group of the stack $\mathcal{X}(\Sigma)$. If $\tau \subseteq \sigma$ is a subcone, we define $\text{Box}(\tau)$ to be the set of elements in $v \in N$ such that $\bar{v} = \sum_{\rho_i \subseteq \tau} a_i \bar{b}_i$, where $0 \leq a_i < 1$. It is easy to see that $\text{Box}(\tau) \subset \text{Box}(\sigma)$. In fact the elements in $\text{Box}(\tau)$ generate a subgroup of the local group $N(\sigma)$. Let $\text{Box}(\Sigma)$ be the union of $\text{Box}(\sigma)$ for all d -dimensional cones $\sigma \in \Sigma$. For $v_1, \dots, v_n \in N$, let $\sigma(\bar{v}_1, \dots, \bar{v}_n)$ be the unique minimal cone in Σ containing $\bar{v}_1, \dots, \bar{v}_n$.

Proposition 5.3 ([BCS]). *The r -th inertia stack of the stack $\mathcal{X}(\Sigma)$ is*

$$\mathcal{I}_r(\mathcal{X}(\Sigma)) = \coprod_{(v_1, \dots, v_r) \in \text{Box}(\Sigma)^r} \mathcal{X}(\Sigma/\sigma(\bar{v}_1, \dots, \bar{v}_r)).$$

We are interested in the cases $r = 1$ or 2 . When $r = 1$,

$$\mathcal{I}(\mathcal{X}(\Sigma)) = \coprod_{v \in \text{Box}(\Sigma)} \mathcal{X}(\Sigma/\sigma(\bar{v}))$$

is the inertia stack. The orbifold Chow ring is the Chow ring of the inertia stack as \mathbb{Z} -modules.

When $r = 2$, for any pair (v_1, v_2) in $\text{Box}(\Sigma)$, there is a unique $v_3 \in \text{Box}(\Sigma)$ such that $v_1 + v_2 + v_3 \equiv 0 \pmod{N}$. We have:

$$(5.7) \quad \mathcal{I}_2(\mathcal{X}(\Sigma)) = \coprod_{(v_1, v_2, v_3); v_1 + v_2 + v_3 \equiv 0 \pmod{N}} \mathcal{X}(\Sigma/\sigma(\bar{v}_1, \bar{v}_2, \bar{v}_3)).$$

The components are called 3-twisted sectors in [CR1].

5.2. The integral orbifold Chow ring. Let $\mathcal{X}(\Sigma)$ be a toric Deligne-Mumford stack with stacky fan Σ and $A_{orb}^*(\mathcal{X}(\Sigma), \mathbb{Z})$ its integral orbifold Chow ring. We first study the $A^*(\mathcal{X}(\Sigma), \mathbb{Z})$ -module structure of $A_{orb}^*(\mathcal{X}(\Sigma), \mathbb{Z})$. Because Σ is a simplicial fan, we have the following two lemmas in [Jiang1]:

Lemma 5.4. *For any $c \in N$, let σ be the minimal cone in Σ containing \bar{c} , then there exists a unique expression*

$$c = v + \sum_{\rho_i \subset \sigma} m_i b_i$$

where $m_i \in \mathbb{Z}_{\geq 0}$, and $v \in \text{Box}(\sigma)$. \square

Lemma 5.5. *Let τ be a cone in the complete simplicial fan Σ and $\{\rho_1, \dots, \rho_s\} \subset \text{link}(\tau)$. Suppose ρ_1, \dots, ρ_s are contained in a cone $\sigma \subset \Sigma$. Then $\sigma \cup \tau$ is contained in a cone of Σ . \square*

Let $v \in \text{Box}(\Sigma)$ and $\sigma := \sigma(\bar{v})$ the minimal cone containing \bar{v} . Then we have the quotient stacky fan Σ/σ and $\Sigma_{\text{red}}/\sigma$. From the diagrams (5.1) and (5.2) we consider the following diagrams:

(5.8)

Taking Gale dual yields

(5.9)

For the quotient stacky fan Σ/σ , if in the map $\beta : \mathbb{Z}^n \rightarrow N$, the vectors $\{b_1, \dots, b_n\}$ generate the torsion part of N , then from (5.8) and (5.9), the vectors $\{\tilde{b}_1, \dots, \tilde{b}_l\}$ in the map $\beta(\sigma) : \mathbb{Z}^l \rightarrow N(\sigma)$ generate the torsion part of $N(\sigma)$. So we have:

Proposition 5.6. *Given a toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ associated to the stacky fan Σ . Suppose that the map β generate the torsion part of N , then for a cone $\sigma \subset \Sigma$, the closed substack $\mathcal{X}(\Sigma/\sigma)$ is a nontrivial gerbe over the toric orbifold $\mathcal{X}(\Sigma_{\text{red}}/\sigma)$. \square*

By Lemma 2.8, the map φ in (5.9) is diagonalizable, so the map $\tilde{\varphi}$ in (5.9) is diagonalizable. From (5.3) and (5.4), $N(\sigma)^\vee \cong \tilde{N}^\vee$, and $\overline{N}(\sigma)^\vee \cong \tilde{\overline{N}}^\vee$, so the map $\varphi(\sigma)$ in (5.9) is diagonalizable. We assume that $\varphi(\sigma)$ is given by the diagonal integral matrix $M(\sigma)$. Let $\tilde{\mathbf{x}}_l = (\tilde{x}_1, \dots, \tilde{x}_l)$ and $\mathbf{x}_l = (x_1, \dots, x_l)$ be column vectors. Then using the same analysis of the formula (3.1), we have the following lemma:

Lemma 5.7. *The formula in (3.1) induces a formula*

$$\tilde{\mathbf{x}}_l = A(\sigma)M(\sigma)C(\sigma)\mathbf{x}_l$$

for the quotient stacky fan, where $A(\sigma)$ and $C(\sigma)$ are integral matrices. When we take x_j as first Chern class of the line bundle L_j , the definition of \tilde{x}_i are compatible with restrictions to components of the inertia stack. \square

Proposition 5.8. *Let $\mathcal{X}(\Sigma)$ be a toric Deligne-Mumford stack associated to the stacky fan Σ , then we have an isomorphism of $A^*(\mathcal{X}(\Sigma), \mathbb{Z})$ -modules:*

$$\bigoplus_{v \in \text{Box}(\Sigma)} A^*(\mathcal{X}(\Sigma/\sigma(\bar{v})), \mathbb{Z}) [\deg(y^v)] \cong \frac{\mathbb{Z}[\Sigma]}{\text{Cir}(\Sigma)}.$$

PROOF. We use a method similar to that in Proposition 4.7 of [Jiang1]. Let

$$S_\Sigma := \mathbb{Z}[y^{b_i} : \rho_i \in \Sigma(1)]/\mathcal{I}_\Sigma.$$

Then $S_\Sigma/\text{Cir}(\Sigma) \cong A^*(\mathcal{X}(\Sigma), \mathbb{Z})$ given by $y^{b_i} \mapsto x_i$. By the definition of $\mathbb{Z}[\Sigma]$ and Lemma 5.4, we see that $\mathbb{Z}[\Sigma] = \bigoplus_{v \in \text{Box}(\Sigma)} y^v \cdot S_\Sigma$. And we obtain an isomorphism of $A^*(\mathcal{X}(\Sigma), \mathbb{Z})$ -modules:

$$(5.10) \quad \frac{\mathbb{Z}[\Sigma]}{\text{Cir}(\Sigma)} \cong \bigoplus_{v \in \text{Box}(\Sigma)} \frac{y^v \cdot S_\Sigma}{y^v \cdot \text{Cir}(\Sigma)}.$$

For any $v \in \text{Box}(\Sigma)$, let $\sigma(\bar{v})$ be the minimal cone in Σ containing \bar{v} . Let $\rho_1, \dots, \rho_l \in \text{link}(\sigma(\bar{v}))$, and $\tilde{\rho}_i$ be the image of ρ_i under the natural map $\pi : N \rightarrow N(\sigma(\bar{v})) = N/N_{\sigma(\bar{v})}$. Then $S_{\Sigma/\sigma(\bar{v})} \subset \mathbb{Z}[\Sigma/\sigma(\bar{v})]$ is the subring generated by: $y^{\tilde{b}_i}$, for $\rho_i \in \text{link}(\sigma(\bar{v}))$. Let \tilde{a} be the order of the torsion subgroup of $N(\sigma(\bar{v}))$. Then let $a = s\tilde{a}$, and conversely we have $\tilde{a} = \frac{1}{s}a$. By Lemmas 5.5 and 5.7, it is easy to check that the ideal $\mathcal{I}_{\Sigma/\sigma(\bar{v})}$ goes to the ideal \mathcal{I}_Σ and we have a morphism $\tilde{\Psi}_v : S_{\Sigma/\sigma(\bar{v})}[\deg(y^v)] \rightarrow y^v \cdot S_\Sigma$ given by: $y^{\tilde{b}_i} \mapsto y^v \cdot sy^{b_i}$. If $\sum_{i=1}^l \tilde{\theta}(\tilde{b}_i) \tilde{a} y^{\tilde{b}_i}$ belongs to the ideal $\text{Cir}(\Sigma/\sigma(\bar{v}))$, then

$$\tilde{\Psi}_v \left(\sum_{i=1}^l \tilde{\theta}(\tilde{b}_i) \tilde{a} y^{\tilde{b}_i} \right) = y^v \cdot \left(\sum_{i=1}^l \theta(b_i) s \tilde{a} y^{b_i} \right) = y^v \cdot \left(\sum_{i=1}^l \theta(b_i) a y^{b_i} \right),$$

where $\theta = \tilde{\theta} \circ \pi$ and $\theta(b_i) = \tilde{\theta}(\tilde{b}_i)$. So we obtain that $\tilde{\Psi}_v(\sum_{i=1}^l \tilde{\theta}(\tilde{b}_i) \tilde{a} y^{\tilde{b}_i}) \in y^v \cdot \text{Cir}(\Sigma)$. So $\tilde{\Psi}_v$ induce a morphism $\Psi_v : \frac{S_{\Sigma/\sigma(\bar{v})}}{\text{Cir}(\Sigma/\sigma(\bar{v}))}[\deg(y^v)] \rightarrow \frac{y^v \cdot S_{\Sigma}}{y^v \cdot \text{Cir}(\Sigma)}$ such that $\Psi_v([y^{\tilde{b}_i}]) = [y^v \cdot s y^{b_i}]$.

Conversely, for such $v \in \text{Box}(\Sigma)$ and $\rho_i \subset \sigma(\bar{v})$, choose $\theta_i \in \text{Hom}_{\mathbb{Z}}(N, \mathbb{Q})$ such that $\theta_i(b_i) = 1$ and $\theta_i(b_{i'}) = 0$ for $b_{i'} \neq b_i \in \sigma(\bar{v})$. Again by Lemmas 5.5 and 5.7 we consider the following morphism $\tilde{\Phi}_v : y^v \cdot S_{\Sigma} \rightarrow S_{\Sigma/\sigma(\bar{v})}[\deg(y^v)]$ given by:

$$y^{b_i} \mapsto \begin{cases} \frac{1}{s} y^{\tilde{b}_i} & \text{if } \rho_i \subseteq \text{link}(\sigma(\bar{v})), \\ -\sum_{j=1}^l \theta_i(b_j) y^{\tilde{b}_j} & \text{if } \rho_i \subseteq \sigma(\bar{v}), \\ 0 & \text{if } \rho_i \not\subseteq \sigma(\bar{v}) \cup \text{link}(\sigma(\bar{v})). \end{cases}$$

Let $y^v \cdot (\sum_{i=1}^n \theta(b_i) a y^{b_i})$ belong to the ideal $y^v \cdot \text{Cir}(\Sigma)$. For $\theta \in M$, we have $\theta = \theta_v + \theta'_v$, where $\theta_v \in N(\sigma(\bar{v}))^* = M \cap \sigma(\bar{v})^\perp$ and θ'_v belongs to the orthogonal complement of the subspace $\sigma(\bar{v})^\perp$ in M . We have

$$\tilde{\Phi}_v \left(y^v \cdot \left(\sum_{i=1}^n \theta(b_i) a y^{b_i} \right) \right) = \sum_{i=1}^l \theta_v(\tilde{b}_i) \tilde{a} y^{\tilde{b}_i} + \sum_{\rho_i \subset \sigma(\bar{v})} \theta'_v(b_i) \left(-\sum_{j=1}^l \theta_i(b_j) y^{\tilde{b}_j} \right) + \sum_{i=1}^l \theta'_v(b_i) y^{\tilde{b}_i}.$$

Note that $\sum_{i=1}^l \theta_v(\tilde{b}_i) \tilde{a} y^{\tilde{b}_i} \in \text{Cir}(\Sigma/\sigma(\bar{v}))$. Now let $\theta'_v = \sum_{\rho_i \subset \sigma(\bar{v})} a_i \theta_i$, where $a_i \in \mathbb{Q}$, then $\sum_{\rho_i \subset \sigma(\bar{v})} \theta'_v(b_i) = \sum_{\rho_i \subset \sigma(\bar{v})} a_i \theta_i(b_i)$. We have:

$$\sum_{\rho_i \subset \sigma(\bar{v})} a_i \theta_i(b_i) \left(-\sum_{j=1}^l \theta_i(b_j) y^{\tilde{b}_j} \right) + \sum_{\rho_i \subset \sigma(\bar{v})} \sum_{j=1}^l a_i \theta_i(b_j) y^{\tilde{b}_j} = 0,$$

so we have $\tilde{\Phi}_v(y^v \cdot (\sum_{i=1}^n \theta(b_i) a y^{b_i})) \in \text{Cir}(\Sigma/\sigma(\bar{v}))$. So $\tilde{\Phi}_v$ induces a morphism

$$\Phi : \frac{y^v \cdot S_{\Sigma}}{y^v \cdot \text{Cir}(\Sigma)} \rightarrow \frac{S_{\Sigma/\sigma(\bar{v})}}{\text{Cir}(\Sigma/\sigma(\bar{v}))}[\deg(y^v)].$$

We check that $\Phi_v \Psi_v = 1$ and $\Psi_v \Phi_v = 1$. So Φ_v is an isomorphism. Note that both sides of (5.10) are $S_{\Sigma}/\text{Cir}(\Sigma) = A^*(\mathcal{X}(\Sigma), \mathbb{Z})$ -modules, we complete the proof. \square

Now we compute the ring structure. The key part of the orbifold cup product is the orbifold obstruction bundle. For the toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$, the obstruction bundle over the 3-twisted sectors in (5.7) is given by:

Proposition 5.9. *Let $\mathcal{X}(\Sigma/\sigma(\bar{v}_1, \bar{v}_2, \bar{v}_3))$ be a 3-twisted sector of the toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$. Let $v_1 + v_2 + v_3 = \sum_{\rho_i \subset \sigma(\bar{v}_1, \bar{v}_2, \bar{v}_3)} a_i b_i$, $a_i = 1, 2$, then the Euler class of the obstruction bundle $O_{(v_1, v_2, v_3)}$ over $\mathcal{X}(\Sigma)_{(v_1, v_2, v_3)}$ is:*

$$\prod_{a_i=2} c_1(\mathcal{L}_i)|_{\mathcal{X}(\Sigma/\sigma(\bar{v}_1, \bar{v}_2, \bar{v}_3))},$$

where \mathcal{L}_i is the line bundle over $\mathcal{X}(\Sigma)$ corresponding to the ray ρ_i .

Proof of Theorem 1.2. Proposition 5.8 gives an isomorphism between $A^*(\mathcal{X}(\Sigma), \mathbb{Z})$ -modules:

$$A_{orb}^*(\mathcal{X}(\Sigma), \mathbb{Z}) \simeq \bigoplus_{v \in \text{Box}(\Sigma)} A^*(\mathcal{X}(\Sigma/\sigma(\bar{v})), \mathbb{Z}) [deg(y^v)] \cong \frac{\mathbb{Z}[\Sigma]}{Cir(\Sigma)}.$$

All we need is to show that the orbifold cup product defined in [AGV] coincides with the product in ring $\mathbb{Z}[\Sigma]/Cir(\Sigma)$. From the above isomorphisms, it suffices to consider the canonical generators y^{b_i}, y^v where $v \in \text{Box}(\Sigma)$.

For any $v_1, v_2 \in \text{Box}(\Sigma)$, let $v_3 \in \text{Box}(\Sigma)$ be the unique box element such that $v_1 + v_2 + v_3 \equiv 0 \pmod{N}$. Then $\mathcal{X}(\Sigma/\sigma(\bar{v}_1, \bar{v}_2, \bar{v}_3))$ is a 3-twisted sector. Let $e_i : \mathcal{X}(\Sigma/\sigma(\bar{v}_1, \bar{v}_2, \bar{v}_3)) \rightarrow \mathcal{X}(\Sigma/\sigma(\bar{v}_i))$ be the evaluation map for $1 \leq i \leq 3$. Let \check{v}_3 be the inverse of v_3 in the local group, and $I : \mathcal{X}(\Sigma/\sigma(\bar{v}_3)) \rightarrow \mathcal{X}(\Sigma/\sigma(\bar{v}_3))$ be the map given by $(x, v_3) \mapsto (x, \check{v}_3)$. Let $\check{e}_3 = I \circ e_3$. Then the orbifold cup product is defined by:

$$y^{v_1} \cup_{orb} y^{v_2} = \check{e}_{3,*}(e_1^* y^{v_1} \cup e_2^* y^{v_2} \cup e(O_{(v_1, v_2, v_3)})),$$

where $O_{(v_1, v_2, v_3)}$ is the obstruction bundle in Proposition 5.9. Since e_1, e_2, \check{e}_3 are all inclusion, so are representable as morphisms of Deligne-Mumford stacks. By [Kr], the pullback and pushforward are well-defined for integral Chow classes. Let $\pi : \mathcal{X}(\Sigma) \rightarrow \mathcal{X}(\Sigma_{\text{red}})$ be the natural morphism of rigidification. The first Chern class of the line bundle L_i is y^{b_i} , so by Proposition 3.3, the first Chern class of \mathcal{L}_i is $\sum_{j=1}^n a_{j,i} y^{b_j}$ which is \tilde{y}^{b_i} . This class represents an integral Chow class of $\mathcal{X}(\Sigma/\sigma(\bar{v}_1, \bar{v}_2, \bar{v}_3))$. We have that $\check{e}_{3,*}(y^{b_i}) = y^{b_i} \in A^*(\mathcal{X}(\Sigma/\sigma(\bar{v}_3)), \mathbb{Z})$. So by the definition of orbifold cup product we have

$$y^{v_1} \cdot y^{v_2} = y^{\check{v}_3} \prod_{i \in I} \tilde{y}^{b_i} \cdot \prod_{i \in J} \tilde{y}^{b_i}.$$

□

6. EXAMPLES

In this section we compute some examples of the integral Chow ring and integral orbifold Chow rings.

Example 6.1 (The moduli stack of 1-pointed elliptic curves). *Let Σ be the complete fan of the projective line, $N = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and $\beta : \mathbb{Z}^2 \rightarrow \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ be given by the vectors $\{b_1 = (2, 1), b_2 = (-3, 0)\}$. Then $\Sigma = (N, \Sigma, \beta)$ is a stacky fan. We compute that $(\beta)^\vee : \mathbb{Z}^2 \rightarrow N^\vee = \mathbb{Z}$ is given by the matrix $[6, 4]$. So we get the following exact sequence:*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^2 \xrightarrow{\beta} \mathbb{Z}^2 \oplus \mathbb{Z}_2 \longrightarrow 0,$$

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^2 \xrightarrow{\beta^\vee} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0,$$

and

$$(6.1) \quad 1 \longrightarrow \mu_2 \longrightarrow \mathbb{C}^* \xrightarrow{[6, 4]^t} (\mathbb{C}^*)^2 \longrightarrow \mathbb{C}^* \longrightarrow 1.$$

The toric Deligne-Mumford stack $\mathcal{X}(\Sigma) = [\mathbb{C}^2 - \{0\}/\mathbb{C}^*] =: \mathbb{P}(6, 4)$, where the action is given by $\lambda(x, y) = (\lambda^6 x, \lambda^4 y)$, may be identified with the moduli stack $\overline{\mathcal{M}}_{1,1}$ of 1-pointed elliptic curves. The stack $\mathcal{X}(\Sigma)$ is the nontrivial μ_2 -gerbe over $\mathbb{P}(3, 2)$ coming from the canonical line bundle over $\mathbb{P}(3, 2)$. Since $N = \mathbb{Z} \oplus \mathbb{Z}_2$, we have $m_i = 2$. By Theorem 1.1, we have

$$A^*(\mathcal{X}(\Sigma), \mathbb{Z}) \cong \frac{\mathbb{Z}[x_1, x_2]}{(2x_1 - 3x_2, 2x_1 2x_2)} \cong \mathbb{Z}[t]/(24t^2),$$

which is the same as the result in [EG]. We compute the integral orbifold Chow ring. There are 7 box elements: $v = (1, 1), w_1 = (-1, 0), w_2 = (-2, 0), u = (0, 1), \rho_1 = (1, 0), \rho_2 = (-1, 1)$ and $\rho_3 = (-2, 1)$ corresponding to 7 twisted sectors. The three box elements v, w_1, u generate the others. So by Theorem 1.2 we have

$$\begin{aligned} A_{\text{orb}}^*(\mathcal{X}(\Sigma), \mathbb{Z}) &\cong \frac{\mathbb{Z}[x_1, x_2, v, w_1, u]}{(2x_1 - 3x_2, 2x_1 2x_2, v^2 - 2x_1 u, w_1^3 - 2x_2 u, v w_1, v 2x_2, w_1 2x_1, u^2 - 1)} \\ &\cong \frac{\mathbb{Z}[t, v, w_1, u]}{(24t^2, v^2 - 6tu, w_1^3 - 4tu, v w_1, 4vt, 6w_1 t, u^2 - 1)}, \end{aligned}$$

which is the same as the result in [AGV].

Example 6.2. In this example we discuss the relation between integral orbifold Chow ring and the integral Chow ring of crepant resolutions. Let $N = \mathbb{Z}^2$, and $\beta : \mathbb{Z}^3 \rightarrow \mathbb{Z}^2$ be given by the vectors $\{b_1 = (1, 0), b_2 = (0, 1), b_3 = (-1, -2)\}$. Let Σ be the complete fan in \mathbb{R}^2 generated by $\{b_1, b_2, b_3\}$. Then $\Sigma = (N, \Sigma, \beta)$ is a stacky fan. We compute that $(\beta)^\vee : \mathbb{Z}^3 \rightarrow N^\vee = \mathbb{Z}$ is given by the matrix $[1, 1, 2]$. So we get the following exact sequence:

$$(6.2) \quad 1 \longrightarrow \mathbb{C}^* \xrightarrow{[1, 1, 2]^t} (\mathbb{C}^*)^3 \longrightarrow (\mathbb{C}^*)^2 \longrightarrow 1$$

The toric Deligne-Mumford stack $\mathcal{X}(\Sigma) = [\mathbb{C}^3 - \{0\}/\mathbb{C}^*]$, where the action is given by $\lambda(x, y, z) = (\lambda x, \lambda y, \lambda^2 z)$, is the weighted projective stack $\mathbb{P}(1, 1, 2)$ which is a toric orbifold. We compute the integral orbifold Chow ring. There is one box element: $v = \frac{1}{2}b_1 + \frac{1}{2}b_3 = (0, 1)$ corresponding to one twisted sector. So by Theorem 1.2 we have

$$\begin{aligned} A_{\text{orb}}^*(\mathcal{X}(\Sigma), \mathbb{Z}) &\cong \frac{\mathbb{Z}[x_1, x_2, x_3, v]}{(x_1 x_2 x_3, x_1 - x_3, x_2 - 2x_3, v^2 - x_1 x_3, v x_1, v x_2, v x_3)} \\ &\cong \frac{\mathbb{Z}[x_3, v]}{(2x_3^3, 2v x_3, v^2 - x_3^2)}. \end{aligned}$$

Let ρ_4 be a ray generated by $v = b_4$. Then the complete fan $\Sigma' = \{b_1, b_2, b_3, b_4\}$ generated by the rays $\{\rho_1, \rho_2, \rho_3, \rho_4\}$ is the fan of Hirzebruch surface \mathbb{F}_2 . It is well-known that

$$\begin{aligned} A^*(\mathbb{F}_2, \mathbb{Z}) &\cong \frac{\mathbb{Z}[x_1, x_2, x_3, x_4]}{(x_1 x_2 x_3, x_1 - x_3, x_2 - 2x_3 - x_4, x_2 x_4, x_1 x_3)} \\ &\cong \frac{\mathbb{Z}[x_3, x_4]}{(2x_3^3 + x_3^2 x_4, x_3^2, 2x_3 x_4 + x_4^2)}. \end{aligned}$$

It is easy to see that these two rings are not isomorphic. So under integer coefficients, $A_{orb}^*(\mathcal{X}(\Sigma), \mathbb{Z}) \not\cong A^*(\mathbb{F}_2, \mathbb{Z})$. In [BMP], the authors proved that

$$A_{orb}^*(\mathcal{X}(\Sigma), \mathbb{C}) \cong A_Q^*(\mathbb{F}_2, \mathbb{C})$$

where $A_Q^*(\mathbb{F}_2, \mathbb{Z})$ is the quantum corrected cohomology of \mathbb{F}_2 under complex number coefficients, thus verifying the cohomological crepant resolution conjecture [R].

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